A NOTE ON ASYMPTOTIC BEHAVIORS OF SOLUTIONS TO QUASILINEAR ELLIPTIC EQUATIONS WITH HARDY POTENTIAL

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Abstract. Optimal estimates on asymptotic behaviors of weak solutions both at the origin and at the infinity are obtained to the following quasilinear elliptic equations

$$-\Delta_p u-\frac{\mu}{|x|^p}|u|^{p-2}u+m|u|^{p-2}u=f(u),\qquad x\in\mathbb{R}^N,$$

where $1<p<N$, $0\le\mu<((N-p)/p)^p$, $m>0$ and f is a continuous function.

Keywords: Quasilinear elliptic equations; Hardy's inequality; Asymptotic behaviors; Comparison principle

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1. Introduction and main results

In this note, we study asymptotic behaviors of weak solutions to the following quasilinear elliptic equations

$$-\Delta_p u-\frac{\mu}{|x|^p}|u|^{p-2}u+m|u|^{p-2}u=f(u),\qquad x\in\mathbb{R}^N,\tag{1.1}$$

where $1<p<N$, $0\le\mu<\bar\mu=((N-p)/p)^p$, $m>0$,

$$\Delta_p u=\sum_{i=1}^N\partial_{x_i}(|\nabla u|^{p-2}\partial_{x_i}u),\qquad \nabla u=(\partial_{x_1}u,\cdots,\partial_{x_N}u),$$

is the p-Laplacian operator and $f:\mathbb{R}\to\mathbb{R}$ is a continuous function denoted by $f\in C(\mathbb{R})$. In addition, we assume throughout the paper that f satisfies that

$$\limsup_{t\to 0}\frac{|f(t)|}{|t|^{q-1}}\le A<\infty\tag{1.2}$$

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for some $q > p$, and that

$$\limsup_{|t|\to\infty} \frac{|f(t)|}{|t|^{p^*-1}} \le A < \infty \tag{1.3}$$

with $p^* = Np/(N-p)$, where $A > 0$ is a constant.

Equation (1.1) is the Euler-Lagrange equation of the energy functional $E : W^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$E(u) = \frac{1}{p}\int_{\mathbb{R}^N} \left(|\nabla u|^p - \frac{\mu}{|x|^p}|u|^p + m|u|^p\right) - \int_{\mathbb{R}^N} F(u), \qquad u \in W^{1,p}(\mathbb{R}^N),$$

where F is given by $F(t) = \int_0^t f$ for $t \in \mathbb{R}$ and $W^{1,p}(\mathbb{R}^N)$ is the Banach space of weakly differentiable functions $u : \mathbb{R}^N \to \mathbb{R}$ such that the norm

$$\|u\|_{1,p,\mathbb{R}^N} = \left(\int_{\mathbb{R}^N} |u|^p\right)^{\frac{1}{p}} + \left(\int_{\mathbb{R}^N} |\nabla u|^p\right)^{\frac{1}{p}}$$

is finite.

All of the integrals in energy functional E are well defined, due to the Sobolev inequality

$$C\left(\int_{\mathbb{R}^N} |\varphi|^{p^*}\right)^{\frac{p}{p^*}} \le \int_{\mathbb{R}^N} |\nabla\varphi|^p, \qquad \forall\, \varphi \in W^{1,p}(\mathbb{R}^N),$$

where $C = C(N,p) > 0$, and due to the Hardy inequality (see [3, Lemma 1.1])

$$\left(\frac{N-p}{p}\right)^p \int_{\mathbb{R}^N} \frac{|\varphi|^p}{|x|^p} \le \int_{\mathbb{R}^N} |\nabla\varphi|^p, \qquad \forall\, \varphi \in W^{1,p}(\mathbb{R}^N), \tag{1.4}$$

and due to the assumptions (1.2) and (1.3), which imply that

$$|F(t)| \le C|t|^p + C|t|^{p^*}, \qquad \forall\, t \in \mathbb{R},$$

for some positive constant C.

We say that $u \in W^{1,p}(\mathbb{R}^N)$ is a weak subsolution of equation (1.1), if for every nonnegative function $\varphi \in C_0^\infty(\mathbb{R}^N)$, the space of smooth functions in $\mathbb{R}^N$ with compact support, there holds

$$\int_{\mathbb{R}^N} \left(|\nabla u|^{p-2}\nabla u \cdot \nabla\varphi - \frac{\mu}{|x|^p}|u|^{p-2}u\varphi + m|u|^{p-2}u\varphi\right) \le \int_{\mathbb{R}^N} f(u)\varphi.$$

A function $u \in W^{1,p}(\mathbb{R}^N)$ is a weak supersolution of equation (1.1), if for every nonnegative function $\varphi \in C_0^\infty(\mathbb{R}^N)$, there holds

$$\int_{\mathbb{R}^N} \left(|\nabla u|^{p-2}\nabla u \cdot \nabla\varphi - \frac{\mu}{|x|^p}|u|^{p-2}u\varphi + m|u|^{p-2}u\varphi\right) \ge \int_{\mathbb{R}^N} f(u)\varphi.$$

A function $u \in W^{1,p}(\mathbb{R}^N)$ is a weak solution of equation (1.1) if it is both a weak subsolution and a weak supersolution.

Equation (1.1) and its variants have been studied extensively. For the existence and the nonexistence of weak solutions to equation (1.1), we refer to e.g. [1, 2, 3]. In this note, we study the asymptotic behaviors of weak solutions to equation (1.1). In the following we study the asymptotic behaviors of positive radial weak solutions and general weak solutions separately.

1.1. **Asymptotic behaviors of positive radial solutions.** In the case when $\mu = 0$, equation (1.1) is reduced to

$$-\Delta_p u + m|u|^{p-2}u = f(u), \qquad \text{in } \mathbb{R}^N. \tag{1.5}$$

When $p = 2$, Gidas, Ni and Nirenberg [5] proved that if u is a positive C^2 solution (not necessarily in $W^{1,2}(\mathbb{R}^N)$) to equation (1.5) satisfying

$$u(x) \to 0 \qquad \text{as } |x| \to \infty, \tag{1.6}$$

and if $f \in C^{1+\alpha}$ for some $\alpha > 0$, then u must be radially symmetric with respect to a point $x_0 \in \mathbb{R}^N$ and

$$\lim_{|x|\to\infty} u(x)|x - x_0|^{\frac{N-1}{2}} e^{\sqrt{m}|x-x_0|} = C \tag{1.7}$$

for a constant $0 < C < \infty$. In fact, the above mentioned result holds under more general assumptions on f. We refer the reader to [5, Theorem 2]. When $1 < p < N$, Li and Zhao [7] proved that if u is a positive radial C^1 distribution solution of equation (1.5) satisfying (1.6), then

$$\lim_{|x|\to\infty} u(x)|x|^{\frac{N-1}{p(p-1)}} e^{\left(\frac{m}{p-1}\right)^{\frac{1}{p}}|x|} = C \tag{1.8}$$

for a constant $0 < C < \infty$.

In the case when $\mu \neq 0$, Deng and Gao [4] studied equation (1.1) with $p = 2$, $m = 1$ and $f(u) = |u|^{\alpha-2}u$, $2 < \alpha < 2^*$, that is,

$$-\Delta u - \frac{\mu}{|x|^2}u + u = |u|^{\alpha-2}u \qquad \text{in } \mathbb{R}^N, \tag{1.9}$$

where $N \geq 3$ and $0 \leq \mu \leq 3\bar{\mu}/4$.

Let $u(x)$ be a positive radial solution to equation (1.9). If u belongs to $W^{1,2}(\mathbb{R}^N)$, Theorem 1.1 of [4] gives the following asymptotic behavior of u at the origin

$$\lim_{|x|\to 0} u(x)|x|^{\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu}} = C, \tag{1.10}$$

for a constant $0 < C < \infty$. Theorem 1.1 of [4] also gives the following asymptotic behavior of u at the infinity

$$\lim_{|x|\to\infty} u(x)|x|^{\frac{N-1}{2}} e^{|x|} = C$$

for a constant $0 < C < \infty$, provided that hypothesis (1.6) holds. For more precise result on the asymptotic behavior of u at infinity, we refer the reader to [4, Theorem 1.1].

Note that Theorem 1.1 of [4] dose not give estimates on the asymptotic behaviors of positive radial solutions to equation (1.9) for $3\bar{\mu}/4 < \mu < \bar{\mu}$. In the general case when $\mu \neq 0$ and $1 < p < N$, the asymptotic behaviors of positive radial solutions to equation (1.1) are either unknown.

In this note, we study the asymptotic behaviors of positive radial weak solutions to equation (1.1) for the full range of parameters p and μ, that is, $1 < p < N$ and $0 \leq \mu < \bar{\mu}$. We have the following estimate for positive radial weak solutions at the origin.

Theorem 1.1. *Assume that $m > 0$, $0 \leq \mu < \bar{\mu} = ((N-p)/p)^p$ and that $f \in C(\mathbb{R})$ satisfies (1.2) and (1.3). Let $u \in W^{1,p}(\mathbb{R}^N)$ be a positive radial weak solution of equation (1.1). Then there exists $\gamma_1 \in [0, (N-p)/p)$ such that*

$$\lim_{|x|\to 0} u(x)|x|^{\gamma_1} = C$$

for a constant $0 < C < \infty$.

We remark that Theorem 1.1 is also true for all $m \in \mathbb{R}$.

In Theorem 1.1 and in the rest of the note, the exponent γ_1 and the exponent γ_2 that will be needed later are defined as follows. Let $\Gamma_\mu : [0,\infty) \to \mathbb{R}$ be defined by

$$\Gamma_\mu(\gamma) \equiv \gamma^{p-1}[(p-1)\gamma - (N-p)] + \mu, \qquad \gamma \in [0,\infty). \tag{1.11}$$

Consider the equation

$$\Gamma_\mu(\gamma) = 0, \qquad \gamma \in [0,\infty). \tag{1.12}$$

Due to our assumptions on N, p and μ, that is, $1 < p < N$, $0 \le \mu < \bar{\mu} = ((N-p)/p)^p$, equation (1.12) admits two and only two nonnegative solutions, which are denoted by γ_1 and γ_2, satisfying

$$0 \le \gamma_1 < \frac{N-p}{p} < \gamma_2 \le \frac{N-p}{p-1}.$$

Note that in the case when $\mu = 0$, we have $\gamma_1 = 0$ and $\gamma_2 = (N-p)/(p-1)$, and that in the case when $p = 2$, we have $\gamma_1 = \sqrt{\bar{\mu}} - \sqrt{\bar{\mu} - \mu}$ and $\gamma_2 = \sqrt{\bar{\mu}} + \sqrt{\bar{\mu} - \mu}$.

As to the asymptotic behavior of positive radial weak solutions of equation (1.1) at the infinity, we follow the argument of Li and Zhao [7] and obtain the following result.

Theorem 1.2. *Assume that $m > 0$, $0 \le \mu < \bar{\mu} = ((N-p)/p)^p$ and that $f \in C(\mathbb{R})$ satisfies (1.2) and (1.3). Let $u \in W^{1,p}(\mathbb{R}^N)$ be a positive radial weak solution of equation (1.1). Then*

$$\lim_{|x|\to\infty} u(x)|x|^{\frac{N-1}{p(p-1)}} e^{\left(\frac{m}{p-1}\right)^{\frac{1}{p}}|x|} = C$$

for a constant $0 < C < \infty$.

In fact, we obtain more precise estimates, see Theorem 2.1 in Section 2.

1.2. **Asymptotic behaviors of general weak solutions.** Now we consider the asymptotic behaviors of general weak solutions to equation (1.1) (not necessarily positive or radially symmetric). Not much is known in this respect.

For radially symmetric weak solution $u \in W^{1,2}(\mathbb{R}^N)$ of equation (1.1) when $p = 2$, it follows from standard argument of ordinary differential equations (see e.g. [1, 11]) that u decays to zero exponentially at infinity (see e.g. [1, Section 4.2] for $p = 2$ and $\mu = 0$). That is, there exist constants $\delta, C > 0$ such that

$$|u(x)| \le Ce^{-\delta|x|}, \qquad \text{for } |x| \text{ large enough.}$$

In general, for $0 \le \mu < \bar{\mu}$ and $1 < p < N$, one can follow the argument of Li [6] to prove that the weak solutions $u \in W^{1,p}(\mathbb{R}^N)$ of equation (1.1) satisfy (1.6).

In the following, we give a complete description on the asymptotic behaviors of general weak solutions to equation (1.1) both at the origin and at the infinity. We have the following result on the asymptotic behavior of general weak solutions at the origin.

Theorem 1.3. *Assume that $m > 0$, $0 \le \mu < \bar{\mu} = ((N-p)/p)^p$ and that $f \in C(\mathbb{R})$ satisfies (1.2) and (1.3). Let $u \in W^{1,p}(\mathbb{R}^N)$ be a weak solution to equation (1.1). Then there exists a positive constant c_1 depending on N, p, μ, m, q, A and the solution u such that*

$$|u(x)| \le c_1|x|^{-\gamma_1} \qquad \text{for } |x| < r_1, \tag{1.13}$$

where r_1, $0 < r_1 < 1$, is a constant depending on N, p, μ, m, q, A and the solution u. If, in addition, both u and $f(u)$ are nonnegative in $B_\rho(0)$ with $\rho > 0$, then there exists a positive constant c_2 depending on N, p, μ, m, q and A such that

$$u(x) \ge c_2 \left(\inf_{B_{r_2}(0)} u\right)|x|^{-\gamma_1} \qquad \text{for } |x| < r_2, \tag{1.14}$$

where r_2, $0 < r_2 < \rho$, is a constant depending on N, p, μ, m, q and A.

We also remark that Theorem 1.3 is true for all $m \in \mathbb{R}$.

In the above Theorem 1.3, the constants c_1 and r_1 depend on the solution u. Precisely, they depend on $\|u\|_{p^*,B_1(0)}$, the L^{p^*}-norm of u in the unit ball $B_1(0)$. They also depend on the modulus of continuity of the function $h(\rho) = \|u\|_{p^*,B_\rho(0)}^{p^*-p}$ at $\rho = 0$ as follows. We can choose a constant $\epsilon_0 > 0$ depending on N, p, μ, m, q and A. Since $h(\rho) \to 0$ as $\rho \to 0$, there exists $\rho_0 > 0$ such that

$$\|u\|_{p^*,B_{\rho_0}(0)}^{p^*-p} < \epsilon_0.$$

The constants c_1, r_1 in Theorem 1.3 depend also on ρ_0.

The estimate (1.13) in Theorem 1.3 follows from the following result proved in [12] (see [12, Theorem 1.3]).

Theorem 1.4. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with $0 \in \Omega$ and let $g \in L^{\frac{N}{p}}(\Omega)$ satisfy*

$$g(x) \le C_0|x|^{-\alpha} \qquad \text{in } \Omega, \tag{1.15}$$

where $C_0 > 0$ and $\alpha < p$. If $u \in W^{1,p}(\Omega)$ is a weak subsolution to equation

$$-\Delta_p w - \frac{\mu}{|x|^p}|w|^{p-2}w = g|w|^{p-2}w \qquad \text{in } \Omega, \tag{1.16}$$

then there exists a constant $C > 0$ depending on N, p, μ, C_0 and α such that

$$u(x) \le CM|x|^{-\gamma_1} \qquad \text{for } x \in B_{r_0}(0),$$

where $M = \sup_{\partial B_{r_0}(0)} u^+$ and r_0, $0 < r_0 < 1$, is a constant depending on N, p, μ, C_0 and α. Here $u^+ = \max(u, 0)$.

Let $u \in W^{1,p}(\mathbb{R}^N)$ be a weak solution to equation (1.1). To apply Theorem 1.4, we set $\Omega = B_1(0)$ and define

$$g(x) = -m + \frac{f(u(x))}{|u(x)|^{p-2}u(x)}. \tag{1.17}$$

Then u is a weak solution to equation (1.16) with function g defined by (1.17). By assumptions (1.2) and (1.3), we have

$$|g(x)| \le C(1 + |u(x)|^{p^*-p}), \tag{1.18}$$

which implies that $g \in L^{\frac{N}{p}}(B_1(0))$ since $u \in L^{p^*}(B_1(0))$ by the Sobolev embedding theorem. Therefore, to apply Theorem 1.4, we only need to verify that g satisfies (1.15) with $C_0 > 0$ and $\alpha < p$. This follows from an apriori estimate for the solution u given by Proposition 3.1. In this way we prove estimate (1.13).

To prove estimate (1.14) in Theorem 1.3, we apply the following comparison principle established in [12] (see [12, Theorem 3.2]).

Theorem 1.5. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $g \in L^{\frac{N}{p}}(\Omega)$. Let $v \in W^{1,p}(\Omega)$ be a weak subsolution to equation (1.16) and $u \in W^{1,p}(\Omega)$ a weak supersolution to equation*

$$-\Delta_p w - \frac{\mu}{|x|^p}|w|^{p-2}w = h|w|^{p-2}w \qquad \text{in } \Omega \tag{1.19}$$

such that $\inf_\Omega u > 0$, where $h \in L^{\frac{N}{p}}(\Omega)$ satisfies $h \ge g$ in Ω. If $v \le u$ on $\partial\Omega$, then we have

$$v \le u \qquad \text{in } \Omega.$$

Let $u \in W^{1,p}(\mathbb{R}^N)$ be a weak solution to equation (1.1) such that u and $f(u)$ are nonnegative in $B_\rho(0)$ with $\rho > 0$. Then u is a nonnegative supersolution to equation

$$-\Delta_p w - \frac{\mu}{|x|^p}|w|^{p-2}w = -m|w|^{p-2}w \tag{1.20}$$

in $B_\rho(0)$. To prove (1.14), we construct a weak subsolution $v \in W^{1,p}(B_{r_2}(0))$ to equation (1.20) in $B_{r_2}(0)$ for some $r_2 \le \rho$, such that $v \le u$ on $\partial B_{r_2}(0)$ and $v \ge C|x|^{-\gamma_1}$ in $B_{r_2}(0)$. Then estimate (1.14) follows from Theorem 1.5.

We also have the following result on the asymptotic behavior of general weak solutions at the infinity.

Theorem 1.6. *Assume that $m > 0$, $0 \le \mu < \bar{\mu} = ((N-p)/p)^p$ and that $f \in C(\mathbb{R})$ satisfies (1.2) and (1.3). Let $u \in W^{1,p}(\mathbb{R}^N)$ be a weak solution to equation (1.1). Then there exists a positive constant C_1 depending on N, p, μ, m, q, A and the solution u such that*

$$|u(x)| \le C_1|x|^{-\frac{N-1}{p(p-1)}} e^{-\left(\frac{m}{p-1}\right)^{\frac{1}{p}}|x|} \qquad \text{for } |x| > R_1, \tag{1.21}$$

where R_1, $R_1 > 1$, is a constant depending on N, p, μ, m, q, A and the solution u. If, in addition, both u and $f(u)$ are nonnegative in $\mathbb{R}^N \backslash B_\rho(0)$ with $\rho > 0$, then there exists a positive constant C_2 depending on N, p, μ, m, q and A such that

$$u(x) \ge C_2 \left(\inf_{\partial B_{R_2}(0)} u\right) |x|^{-\frac{N-1}{p(p-1)}} e^{-\left(\frac{m}{p-1}\right)^{\frac{1}{p}}|x|} \qquad \text{for } |x| > R_2, \tag{1.22}$$

where R_2, $R_2 > \rho$, is a constant depending on N, p, μ, m, q and A.

We also prove Theorem 1.6 by the comparison principle. We prove (1.22) as follows. We can prove (1.21) in a similar way. Let $u \in W^{1,p}(\mathbb{R}^N)$ be a weak solution to equation (1.1) such that u and $f(u)$ are nonnegative in $\mathbb{R}^N \backslash B_\rho(0)$ with $\rho > 0$. Then u is a supersolution to equation

$$-\Delta_p w + m|w|^{p-2}w = 0 \qquad \text{in } \mathbb{R}^N \backslash B_\rho(0). \tag{1.23}$$

We construct a subsolution v to equation (1.23) such that $v(x) \le u(x)$ on $\partial B_\rho(0)$ and that $v(x) \ge C|x|^{-\frac{N-1}{p(p-1)}} e^{-\left(\frac{m}{p-1}\right)^{\frac{1}{p}}|x|}$ in $\mathbb{R}^N \backslash B_\rho(0)$. Then it follows from the comparison principle that $u \ge v$ in $\mathbb{R}^N \backslash B_\rho(0)$, which proves (1.22).

The paper is organized as follows. We prove Theorem 1.1 and Theorem 1.2 in Section 2, Theorem 1.3 in Section 3 and Theorem 1.6 in Section 4.

Our notations are standard. $B_R(x)$ is the open ball in $\mathbb{R}^N$ centered at x with radius $R > 0$ and $B_R^c(x) = \mathbb{R}^N \backslash B_R(x)$. We write

$$\fint_E u = \frac{1}{|E|}\int_E u,$$

whenever $E \subset \mathbb{R}^N$ is a Lebesgue measurable set and $|E|$, the n-dimensional Lebesgue measure of set E, is positive and finite. Let Ω be an arbitrary domain in $\mathbb{R}^N$. We denote by $C_0^\infty(\Omega)$ the space of smooth functions with compact support in Ω. For any $1 \le s \le \infty$, $L^s(\Omega)$ is the Banach space of Lebesgue measurable functions u such that the norm

$$\|u\|_{s,\Omega} = \begin{cases} \left(\int_\Omega |u|^s\right)^{\frac{1}{s}} & \text{if } 1 \le s < \infty \\ \operatorname{esssup}_\Omega |u| & \text{if } s = \infty \end{cases}$$

is finite. A function u belongs to the Sobolev space $W^{1,s}(\Omega)$ if $u \in L^s(\Omega)$ and its first order weak partial derivatives also belong to $L^s(\Omega)$. We endow $W^{1,s}(\Omega)$ with the norm

$$\|u\|_{1,s,\Omega} = \|u\|_{s,\Omega} + \|\nabla u\|_{s,\Omega}.$$

For the properties of the Sobolev functions, we refer to the monograph [13]. By abuse of notation, if u is a radially symmetric function in $\mathbb{R}^N$, we write $u(x) = u(r)$ with $r = |x|$.

2. Proofs of Theorem 1.1 and Theorem 1.2

We prove Theorem 1.1 and Theorem 1.2 in this section. In the case when $\mu = 0$, Theorem 1.1 can be proved easily, and Theorem 1.2 was proved in [7]. So in this section we always assume that $0 < \mu < \bar{\mu}$.

Let $u \in W^{1,p}(\mathbb{R}^N)$ be a positive radial weak solution of equation (1.1). By abuse of notation, we write $u(x) = u(r)$ with $r = |x|$. Then since $u \in W^{1,p}(\mathbb{R}^N)$, we have

$$\int_0^\infty \left(|u(r)|^p + |u'(r)|^p\right) r^{N-1} = \frac{1}{\omega_{N-1}} \int_{\mathbb{R}^N} \left(|u|^p + |\nabla u|^p\right) < \infty, \tag{2.1}$$

where ω_{N-1} is the surface measure of the unit sphere in $\mathbb{R}^N$, and u is a weak solution to the following ordinary differential equation

$$\begin{cases} -\left(r^{N-1}|u'|^{p-2}u'\right)' = r^{N-1}\left(\dfrac{\mu}{r^p}u^{p-1} - mu^{p-1} + f(u)\right), & r > 0, \\ u(r) > 0 \quad \text{for } r > 0. \end{cases} \tag{2.2}$$

Before proving Theorem 1.1 and Theorem 1.2, we remark that in fact both u and $r^{N-1}|u'|^{p-2}u'$ are continuously differentiable in $(0, \infty)$, and equation (2.2) can be understood in the classical sense. Indeed, it is well known that every radially symmetric function in $W^{1,p}(\mathbb{R}^N)$, after modifying on a set of measure zero, is a continuous function in $(0, \infty)$. Then by the continuity of f, we deduce that $r^{N-1}\left(\frac{\mu}{r^p}u^{p-1} - mu^{p-1} + f(u)\right) \in C(0, \infty)$, which implies by equation (2.2) that $r^{N-1}|u'|^{p-2}u' \in C^1(0, \infty)$. Thus equation (2.2) can be understood in the classical sense.

2.1. Proof of Theorem 1.1.

We prove Theorem 1.1 now. We start the proof by claiming that

$$u'(r) < 0 \qquad \text{for } r\text{sufficiently small}. \tag{2.3}$$

Indeed, note that since $u \in W^{1,p}(\mathbb{R}^N)$ is a radial function, we have by [9, Corollary II.1] that

$$u(r) r^{\frac{N-p}{p}} = o(1) \qquad \text{as } r \to 0.$$

Then by (1.2), (1.3) and the above estimate, we have

$$\left|\frac{f(u(r))r^p}{u^{p-1}(r)}\right| \le Cr^p\left(1 + u^{p^*-p}(r)\right) = o(1) \qquad \text{as } r \to 0. \tag{2.4}$$

Hence

$$\frac{\mu}{r^p} - m + \frac{f(u)}{u^{p-1}} = \frac{1}{r^p}\left(\mu - mr^p + \frac{f(u(r))r^p}{u^{p-1}(r)}\right) > \frac{\mu}{2r^p} > 0 \qquad \text{for } r \text{ small enough}.$$

Therefore $\left(r^{N-1}|u'|^{p-2}u'\right)' < 0$ for r small enough by equation (2.2). Hence $r^{N-1}|u'|^{p-2}u'$ is strictly decreasing for r small enough. So we can assume that $\lim_{r\to 0} r^{N-1}|u'|^{p-2}u' = a$ for some $a \in (-\infty, \infty]$. We will prove that $a = 0$. Suppose, on the contrary, that $a \neq 0$. Then there exist constants $C, r_0 > 0$ such that $|u'(r)| \ge Cr^{-\frac{N-1}{p-1}}$ for $0 < r < r_0$. Then we have

$$\int_0^{r_0} |u'(r)|^p r^{N-1} \ge C\int_0^{r_0} r^{-\frac{N-1}{p-1}} = \infty.$$

We reach a contradiction to (2.1). Hence $a = 0$. Therefore $r^{N-1}|u'|^{p-2}u' < 0$ for r small enough. This proves (2.3).

Consider the function

$$w(r) = -\frac{r^{p-1}|u'(r)|^{p-2}u'(r)}{u^{p-1}(r)} \qquad \text{for } r > 0. \tag{2.5}$$

Then $w \in C^1(0,\infty)$, $w(r)>0$ for $r>0$ small enough by (2.3), and w satisfies

$$w'(r) = \frac{1}{r}\Gamma_\mu\left(w^{\frac{1}{p-1}}(r)\right) + r^{p-1}\left(-m + \frac{f(u)}{u^{p-1}}\right). \tag{2.6}$$

Recall that Γ_μ is defined as in (1.11). To prove Theorem 1.1, it is enough to prove that

$$w(r) = \gamma_1^{p-1} + o(r^\delta), \qquad \text{as } r \to 0, \tag{2.7}$$

for some $\delta \in (0,1)$.

First, we prove that $\lim_{r\to 0} w(r)$ exists and

$$\lim_{r\to 0} w(r) = \gamma_1^{p-1}. \tag{2.8}$$

To prove that $\lim_{r\to 0} w(r)$ exists, we suppose, on the contrary, that

$$\beta \equiv \limsup_{r\to 0} w > \liminf_{r\to 0} w \equiv \alpha.$$

Then there exist two sequences of positive numbers $\{\xi_i\}$ and $\{\eta_i\}$ such that $\xi_i \to 0$ and $\eta_i \to 0$ and that $\eta_i > \xi_i > \eta_{i+1}$ for all $i = 1,2,\cdots$. Moreover, the function w has a local maximum at ξ_i and a local minimum at η_i for all $i=1,2,\cdots$, and

$$\lim_{i\to\infty} w(\xi_i) = \beta, \qquad \lim_{i\to\infty} w(\eta_i) = \alpha.$$

Note that $w'(\xi_i) = w'(\eta_i) = 0$. By equation (2.6), we have that

$$\Gamma_\mu\left(w^{\frac{1}{p-1}}(\xi_i)\right) - m\xi_i^p + \frac{f(u(\xi_i))\xi_i^p}{u^{p-1}(\xi_i)} = 0,$$

and that

$$\Gamma_\mu\left(w^{\frac{1}{p-1}}(\eta_i)\right) - m\eta_i^p + \frac{f(u(\eta_i))\eta_i^p}{u^{p-1}(\eta_i)} = 0.$$

By (2.4) and the above two equalities,

$$\lim_{i\to\infty} \Gamma_\mu\left(w^{\frac{1}{p-1}}(\xi_i)\right) = \lim_{i\to\infty} \Gamma_\mu\left(w^{\frac{1}{p-1}}(\eta_i)\right) = 0.$$

Since $\Gamma_\mu(s) \to \infty$ as $s\to\infty$, $\{w(\xi_i)\}$ and $\{w(\eta_i)\}$ are bounded. So α,β are finite and

$$\Gamma_\mu\left(\beta^{\frac{1}{p-1}}\right) = \Gamma_\mu\left(\alpha^{\frac{1}{p-1}}\right) = 0.$$

Recall that $\Gamma_\mu(\gamma)=0$ if and only if $\gamma=\gamma_1$ or $\gamma=\gamma_2$. Recall also that $\gamma_1<\gamma_2$ (see (1.12) for the definition of γ_1 and γ_2). Hence

$$\beta = \gamma_2^{p-1} \quad \text{and} \quad \alpha = \gamma_1^{p-1}.$$

That is,

$$\lim_{i\to\infty} w(\xi_i) = \gamma_2^{p-1} \quad \text{and} \quad \lim_{i\to\infty} w(\eta_i) = \gamma_1^{p-1}.$$

Note that $\gamma_1 < (N-p)/p < \gamma_2$. So there exists $\zeta_i \in (\eta_{i+1}, \xi_i)$ such that

$$w(\eta_{i+1}) < w(\zeta_i) = \left(\frac{N-p}{p}\right)^{p-1} < w(\xi_i)$$

for i large enough. Then by (2.4) and equation (2.6), we obtain that

$$\zeta_i w'(\zeta_i) = \Gamma_\mu\left(\frac{N-p}{p}\right) - m\zeta_i^p + \frac{f(u(\zeta_i))\zeta_i^p}{u^{p-1}(\zeta_i)} = -(\bar\mu - \mu) + o(1) < 0$$

for i large enough. Here we used the fact that

$$\Gamma_\mu\left(\frac{N-p}{p}\right) = -(\bar\mu - \mu).$$

Hence $w'(\zeta_i) < 0$ for i large enough. Therefore w is strictly decreasing in a neighborhood of ζ_i. Since $\zeta_i < \xi_i$ and $w(\zeta_i) < w(\xi_i)$, there exists $\zeta_i < \zeta_i' < \xi_i$ such that $w(r) \le w(\zeta_i)$ for $\zeta_i < r < \zeta_i'$ and $w(\zeta_i') = w(\zeta_i)$. Thus $w'(\zeta_i') \ge 0$. However, by equation (2.6), we have that $w'(\zeta_i') < 0$. We reach a contradiction. Therefore $\lim_{r\to 0} w(r)$ exists.

Set $k^{p-1} = \lim_{r\to 0} w(r)$. We will prove that $k = \gamma_1$.

We claim that $k \le (N-p)/p$. Otherwise, choose $\epsilon > 0$ such that $k - \epsilon > (N-p)/p$. Then for r small enough we have $w(r) > (k-\epsilon)^{p-1}$, that is, $-ru'(r)/u(r) > k - \epsilon$ for r small enough. This implies that $u(r) \ge Cr^{\epsilon - k}$ for r small enough, which implies $u \notin L^{p^*}(B_1(0))$. We reach a contradiction. Thus $k \le (N-p)/p$.

By (2.4) and equation (2.6), we have that

$$\lim_{r\to 0} rw'(r) = \Gamma_\mu(k).$$

We claim that $\Gamma_\mu(k) = 0$. Otherwise, suppose that $\Gamma_\mu(k) \neq 0$. Note that for any $0 < s < s_0$, we have

$$w(s_0) = w(s) + \int_s^{s_0} w'.$$

Then $\Gamma_\mu(k) \neq 0$ implies that $\lim_{s\to 0} \left|\int_s^{s_0} w'\right| = \infty$ if s_0 is small enough. This contradicts to (2.8). Hence $\Gamma_\mu(k) = 0$. Recall that $\Gamma_\mu(\gamma) = 0$ if and only if $\gamma = \gamma_1$ or $\gamma = \gamma_2$. Thus we have either $k = \gamma_1$ or $k = \gamma_2$. Then we deduce that $k = \gamma_1$ since $k \le (N-p)/p < \gamma_2$. This proves (2.8).

As a result, (2.8) implies that for any $\epsilon > 0$ sufficiently small there exist $C, C' > 0$ such that

$$C' r^{-\gamma_1 + \epsilon} \le u(r) \le C r^{-\gamma_1 - \epsilon}$$

for $r > 0$ small enough. Choose $\epsilon = \epsilon_0 > 0$ such that $p - (p^* - p)(\gamma_1 + \epsilon_0) > 0$. Applying (2.4), we obtain that

$$\left|\frac{f(u(r))r^p}{u^{p-1}(r)}\right| \le Cr^p \left(1 + u^{p^*-p}(r)\right) \le Cr^{p-(p^*-p)(\gamma_1+\epsilon_0)} \equiv Cr^{\delta_0} \tag{2.9}$$

for $r > 0$ small enough. Here $\delta_0 \equiv p - (p^* - p)(\gamma_1 + \epsilon_0) > 0$.

Now we prove (2.7). Let $w_1(r) = w(r) - \gamma_1^{p-1}$. Then $w_1(r) \to 0$ as $r \to 0$. We prove that $w_1(r) = o(r^\delta)$ as $r \to 0$ for some $\delta > 0$.

By equation (2.6) and the definition of Γ_μ (see (1.11)), we have

$$\begin{aligned} w_1'(r) = w'(r) &= \frac{1}{r}\Gamma_\mu\left(w^{\frac{1}{p-1}}(r)\right) + r^{p-1}\left(-m + \frac{f(u(r))}{u^{p-1}(r)}\right) \\ &= \frac{1}{r}\left((p-1)w^{\frac{p}{p-1}}(r) - (N-p)w(r) + \mu\right) - mr^{p-1} + \frac{f(u(r))r^{p-1}}{u^{p-1}(r)} \\ &= \frac{A(r)}{r}w_1(r) + B(r), \end{aligned} \tag{2.10}$$

for r small enough, where $A(r) \to p\gamma_1 - (N-p) < 0$ as $r \to 0$ and

$$B(r) = -mr^{p-1} + \frac{f(u(r))r^{p-1}}{u^{p-1}(r)} = O\left(r^{\delta_0 - 1}\right) \qquad \text{as } r \to 0, \tag{2.11}$$

by (2.9). Here $\delta_0 > 0$ is as in (2.9).

Fix $r_0 > 0$ small and define $h(r) = \int_r^{r_0} A(\tau)\tau^{-1}d\tau$ for $0 < r < r_0$. Since w_1 is a solution to equation (2.10), it has the following form

$$w_1(r) = \int_0^r e^{h(s)-h(r)}B(s)ds.$$

Since $h(s)-h(r)=\int_s^r A(\tau)\tau^{-1}d\tau<0$ for $0<s<r$, we obtain that $e^{h(s)-h(r)}\le 1$ for $0<s<r$. Hence by (2.11), we have for r small enough that

$$|w_1(r)|\le\int_0^r|B|\le Cr^{\delta_0}.$$

Here $\delta_0>0$ is as in (2.9). This proves (2.7).

Recall that w is defined as (2.5). The conclusion of Theorem 1.1 follows easily from estimate (2.7). The proof of Theorem 1.1 is complete.

We remark here that the proof of Theorem 1.1 above works for all $m\in\mathbb{R}$.

2.2. **Proof of Theorem 1.2.** Following the argument of Li and Zhao [7], we have the following more precise result which implies Theorem 1.2.

Theorem 2.1. *Assume that $m>0$, $0\le\mu<\bar{\mu}=((N-p)/p)^p$ and that $f\in C(\mathbb{R})$ satisfies (1.2). Let $u\in W^{1,p}(\mathbb{R}^N)$ be a positive radial weak solution to equation (1.1) and let k be the integer such that $k\le p<k+1$. Then $u'(r)<0$ for r large enough, and*

$$\left(-\frac{u'(r)}{u(r)}\right)^{p-1}=\sum_{i=0}^k\frac{c_i}{r^i}-\frac{((p-1)/m)^{1/p}\mu}{pr^p}+O\left(\frac{1}{r^{k+1}}\right)\qquad \text{as } r\to\infty, \tag{2.12}$$

where

$$c_0=\left(\frac{m}{p-1}\right)^{\frac{p-1}{p}},\qquad c_1=\frac{N-1}{p}\left(\frac{m}{p-1}\right)^{\frac{p-2}{p}},$$

and $\{c_i\}_{i=2}^k$ are determined uniquely by

$$(N-i)c_{i-1}-p\left(\frac{m}{p-1}\right)^{\frac{1}{p}}c_i=\sum_{n=2}^i\frac{F^{(n)}(0)}{n!}\sum_{\substack{j_1+\cdots+j_n=i\\ j_1,\cdots,j_n>0}}c_{j_1}c_{j_2}\cdots c_{j_n},$$

where $F^{(n)}(0)$ is the n-th derivative of the function $F(t)=(p-1)(c_0+t)^{\frac{p}{p-1}}$ at $t=0$.

We remark that $u(r)\to 0$ as $r\to\infty$ since $u\in W^{1,p}(\mathbb{R}^N)$ is a radially symmetric function. We follow the argument of Li and Zhao [7] to prove Theorem 2.1, with some modifications.

Proof of Theorem 2.1. We start the proof by claiming that

$$u'(r)<0\qquad \text{for } r \text{ large enough.} \tag{2.13}$$

Indeed, we have by (1.6) and (1.2) that

$$\frac{\mu}{r^p}u^{p-1}-mu^{p-1}+f(u)=u^{p-1}\left(\frac{\mu}{r^p}-m+\frac{f(u)}{u^{p-1}}\right)\le-\frac{m}{2}u^{p-1}<0$$

for r large. Hence $\left(r^{N-1}|u'|^{p-2}u'\right)'>0$ for r large. Thus $r^{N-1}|u'|^{p-2}u'$ increases to a limit $l\le\infty$ as $r\to\infty$. We prove that $l\le 0$. Otherwise, if $l>0$, then $u'(r)>0$ for r large enough. Since $u(r)\to 0$ as $r\to 0$, we have $u(r)<0$ for r large enough. We obtain a contradiction, since we assume that u is a positive solution in the theorem. Hence $l\le 0$, and then $r^{N-1}|u'|^{p-2}u'<0$ for r large. This proves (2.13).

Now consider the function

$$\phi(r)=-\frac{|u'(r)|^{p-2}u'(r)}{u^{p-1}(r)}\qquad\text{for } r>0.$$

Then $\phi\in C^1(0,\infty)$, $\phi(r)>0$ for r large enough by (2.13) and ϕ satisfies the equation

$$\phi'=(p-1)\phi^{\frac{p}{p-1}}-\frac{N-1}{r}\phi+\frac{\mu}{r^p}-m+\frac{f(u)}{u^{p-1}}. \tag{2.14}$$

It follows from (1.2) that $f(u)/u^{p-1} = O(u^\delta)$ as $r \to \infty$. Here $\delta = q - p > 0$.

We study the asymptotic behavior of ϕ at the infinity. First, we claim that

$$\limsup_{r\to\infty} \phi(r) < \infty. \tag{2.15}$$

Indeed, note that by Young's inequality,

$$\frac{N-1}{r}\phi \le \frac{p-1}{2}\phi^{\frac{p}{p-1}} + \frac{C_{N,p}}{r^p}$$

for a constant $C_{N,p} > 0$. Hence by (2.14) we have

$$\phi'(r) \ge \frac{p-1}{2}\phi^{\frac{p}{p-1}}(r) - m + \frac{\mu - C_{N,p}}{r^p} + \frac{f(u(r))}{u^{p-1}(r)}. \tag{2.16}$$

Note that

$$\frac{\mu - C_{N,p}}{r^p} + \frac{f(u(r))}{u^{p-1}(r)} \to 0 \qquad \text{as } r \to \infty.$$

Hence there is a constant $K > 0$ such that by (2.16), we have

$$\phi'(r) \ge \frac{p-1}{2}\phi^{\frac{p}{p-1}}(r) - K \tag{2.17}$$

for r large enough. Suppose, on the contrary, that $\limsup_{r\to\infty} \phi(r) = \infty$. Then there exists $r_0 > 1$ large enough such that $\phi(r_0) > \left(\frac{4K}{p-1}\right)^{\frac{p-1}{p}}$, that is, $\frac{p-1}{2}\phi^{\frac{p}{p-1}}(r_0) > 2K$. Then we have $\phi'(r_0) \ge K > 0$. This implies that ϕ is an increasing function in a neighborhood of r_0. Hence there exists $\epsilon > 0$ such that $\phi(s) \ge \phi(r_0)$ for all $s \in [r_0, r_0 + \epsilon]$. Let

$$r_1 = \sup\left\{r;\ r > r_0 \text{ and } \phi(s) \ge \phi(r_0) \text{ for all } s \in [r_0, r]\right\}.$$

Then $r_1 \ge r_0 + \epsilon$. We prove that $r_1 = \infty$. Otherwise, suppose that $r_1 < \infty$. Then we have that $\phi(s) \ge \phi(r_0)$ for all $s \in [r_0, r_1]$ and $\phi(r_1) = \phi(r_0)$. This implies that $\phi'(r_1) \le 0$. However, by (2.17), we have

$$\phi'(r_1) \ge \frac{p-1}{2}\phi^{\frac{p}{p-1}}(r_1) - K = \frac{p-1}{2}\phi^{\frac{p}{p-1}}(r_0) - K \ge K > 0.$$

We reach a contradiction. Hence $r_1 = \infty$. That is, $\phi(r) \ge \phi(r_0) > \left(\frac{4K}{p-1}\right)^{\frac{p-1}{p}}$ for all $r \ge r_0$. Then we can deduce from (2.17) that

$$\phi'(r) \ge \frac{p-1}{4}\phi^{\frac{p}{p-1}}(r) \qquad \text{for } r > r_0. \tag{2.18}$$

Solving equation (2.18) gives us a number $r_2 = 4\phi^{-1/(p-1)}(r_0) + r_0 < \infty$ such that

$$\phi(r) \ge \left(\frac{4}{r_2 - r}\right)^{p-1} \qquad \text{for } r_0 < r < r_2.$$

Thus $\phi(r_2) = \lim_{r\uparrow r_2} \phi(r) = \infty$. We reach a contradiction. Hence $\limsup_{r\to\infty} \phi(r) < \infty$. This proves (2.15).

Second, we claim that

$$\lim_{r\to\infty} \phi(r) = \left(\frac{m}{p-1}\right)^{\frac{p-1}{p}} := \phi_\infty. \tag{2.19}$$

To prove that $\lim_{r\to\infty} \phi(r)$ exists, we suppose on the contrary that

$$\beta \equiv \limsup_{r\to\infty} \phi(r) > \liminf_{r\to\infty} \phi(r) \equiv \alpha.$$

Then $\beta < \infty$ by (2.15) and there exist two sequences of positive numbers $\{\xi_i\}$ and $\{\eta_i\}$ such that $\xi_i \to \infty$ and $\eta_i \to \infty$. Moreover, the function ϕ has a local maximum at ξ_i and a local minimum at η_i for all $i = 1, 2, \cdots$, and

$$\lim_{i\to\infty} \phi(\xi_i) = \beta, \qquad \lim_{i\to\infty} \phi(\eta_i) = \alpha.$$

Note that $\phi'(\xi_i) = \phi'(\eta_i) = 0$. By equation (2.14), we have that

$$(p-1)\phi^{\frac{p}{p-1}}(\xi_i) - \frac{N-1}{\xi_i}\phi(\xi_i) + \frac{\mu}{\xi_i^p} - m + \frac{f(u(\xi_i))}{u^{p-1}(\xi_i)} = 0,$$

and that

$$(p-1)\phi^{\frac{p}{p-1}}(\eta_i) - \frac{N-1}{\eta_i}\phi(\eta_i) + \frac{\mu}{\eta_i^p} - m + \frac{f(u(\eta_i))}{u^{p-1}(\eta_i)} = 0.$$

Letting $i \to \infty$, we obtain that

$$(p-1)\beta^{\frac{p}{p-1}} - m = 0 \quad \text{and} \quad (p-1)\alpha^{\frac{p}{p-1}} - m = 0.$$

That is, $\alpha = \beta = \left(\frac{m}{p-1}\right)^{\frac{p-1}{p}}$. We reach a contradiction. Thus $\lim_{r\to\infty} \phi(r)$ exists.

Set $\phi_\infty = \lim_{r\to\infty} \phi(r)$. Then $\phi_\infty \geq 0$. By (2.15) we have $\phi_\infty < \infty$. Letting $r \to \infty$ in equation (2.14) yields that $\lim_{r\to\infty} \phi'(r) = (p-1)\phi_\infty^{\frac{p}{p-1}} - m$. We claim that $(p-1)\phi_\infty^{\frac{p}{p-1}} - m = 0$. Otherwise, suppose that $(p-1)\phi_\infty^{\frac{p}{p-1}} - m \neq 0$. Note that for any $r > s$ we have

$$\phi(r) = \phi(s) + \int_s^r \phi'.$$

Then $(p-1)\phi_\infty^{\frac{p}{p-1}} - m \neq 0$ implies that $\lim_{r\to\infty} |\int_s^r \phi'| = \infty$. We reach a contradiction to (2.15). Thus $(p-1)\phi_\infty^{\frac{p}{p-1}} - m = 0$. This proves (2.19).

By (2.19), we deduce that

$$\lim_{r\to\infty} \frac{u'}{u} = \lim_{r\to\infty} \left(-\phi^{\frac{1}{p-1}}\right) = -\left(\frac{m}{p-1}\right)^{\frac{1}{p}}.$$

Therefore for any $m > \epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that

$$u(r) \leq C_\epsilon e^{-\left(\frac{m-\epsilon}{p-1}\right)^{\frac{1}{p}} r}$$

for r large enough. Take $\epsilon = m/2$ and set $\delta_1 = \left(\frac{m}{2(p-1)}\right)^{\frac{1}{p}}$. Then

$$u(r) \leq Ce^{-\delta_1 r} \qquad \text{for } r \text{ large enough.} \tag{2.20}$$

Next, we give a precise expansion of $\phi(r)$ at infinity. Let $\phi_1 = \phi - \phi_\infty$ and $F(t) = (p-1)(t + \phi_\infty)^{p/(p-1)}$ for $t \geq 0$. Equation (2.14) gives

$$\begin{aligned}
&\phi_1' - \alpha_0\phi_1 + \frac{N-1}{r}\phi_1 \\
=&F(\phi_1) - F(0) - F'(0)\phi_1 - \frac{(N-1)\,\phi_\infty}{r} + \frac{\mu}{r^p} + O(u^\delta),
\end{aligned} \tag{2.21}$$

where $\alpha_0 = p\phi_\infty^{1/(p-1)}$. Note that $F(0) = m$ and $F'(0) = \alpha_0$. Since $\phi_1(r) \to 0$ as $r \to \infty$, we have

$$F(\phi_1) - F(0) - F'(0)\phi_1 = O(\phi_1^2)$$

as $r \to \infty$. Thus (2.21) is reduced to

$$\phi_1' - \alpha_0\phi_1 + \frac{N-1}{r}\phi_1 = -\frac{(N-1)\,\phi_\infty}{r} + \frac{\mu}{r^p} + O(u^\delta) + O(\phi_1^2). \tag{2.22}$$

Multiply both sides of equation (2.22) by ϕ_1. We have that

$$\frac{1}{2}\phi_1^2(r)+\int_r^\infty\left(\alpha_0-\frac{N-1}{s}+O(\phi_1)\right)\phi_1^2=\int_r^\infty\left(\frac{(N-1)\,\phi_\infty}{s}-\frac{\mu}{s^p}\right)\phi_1-\int_r^\infty O(u^\delta)\phi_1.$$

We can take r sufficiently large such that

$$\alpha_0-\frac{N-1}{s}+O(\phi_1)\geq\frac{\alpha_0}{2}\text{ and }\frac{(N-1)\,\phi_\infty}{s}\geq\frac{\mu}{s^p}\quad\text{ for }s\geq r.$$

Then

$$\phi_1^2(r)+\alpha_0\int_r^\infty\phi_1^2\leq 2\int_r^\infty\frac{(N-1)\,\phi_\infty}{s}|\phi_1|-2\int_r^\infty O(u^\delta)\phi_1.$$

Note that

$$2\int_r^\infty\frac{(N-1)\,\phi_\infty}{s}|\phi_1|\leq\frac{\alpha_0}{4}\int_r^\infty\phi_1^2+\frac{4\,(N-1)^2\,\phi_\infty^2}{\alpha_0}\int_r^\infty\frac{1}{s^2}$$

and

$$2\int_r^\infty O(u^\delta)\phi_1\leq\frac{\alpha_0}{4}\int_r^\infty\phi_1^2+\frac{4}{\alpha_0}\int_r^\infty O(u^{2\delta}).$$

By virtue of the above two inequalities and (2.20), we obtain for sufficiently large r that

$$\phi_1^2(r)+\frac{\alpha_0}{2}\int_r^\infty\phi_1^2\leq\frac{4\,(N-1)^2\,\phi_\infty^2}{\alpha_0}\frac{1}{r}+Ce^{-2\delta\delta_1 r}\leq\frac{8\,(N-1)^2\,\phi_\infty^2}{\alpha_0}\frac{1}{r}.$$

Thus we have

$$\phi_1^2(r)=O\left(\frac{1}{r}\right)\qquad\text{as }r\to\infty.\tag{2.23}$$

Combining (2.23) and (2.22) gives us

$$\phi_1'(r)-\alpha_0\phi_1+\frac{N-1}{r}\phi_1=O\left(\frac{1}{r}\right)\qquad\text{as }r\to\infty.\tag{2.24}$$

Therefore we get from (2.24) for r sufficiently large that

$$\left(r^{N-1}e^{-\alpha_0 r}\phi_1(r)\right)'=r^{N-1}e^{-\alpha_0 r}O\left(\frac{1}{r}\right).\tag{2.25}$$

Integrate both sides of (2.25). We obtain that for r sufficiently large,

$$\phi_1(r)=\frac{e^{\alpha_0 r}}{r^{N-1}}\int_r^\infty s^{N-1}e^{-\alpha_0 s}O\left(\frac{1}{s}\right).\tag{2.26}$$

Then it follows from (2.26) that

$$\phi_1(r)=O\left(\frac{1}{r}\right)\qquad\text{as }r\to\infty,\tag{2.27}$$

which is an improvement of (2.23). Using (2.27) and (2.22) we obtain for r sufficiently large that

$$\left(r^{N-1}e^{-\alpha_0 r}\phi_1(r)\right)'=\frac{e^{\alpha_0 r}}{r^{N-1}}\left(-\frac{(N-1)\,\phi_\infty}{r}+\frac{\mu}{r^p}+O\left(\frac{1}{r^2}\right)\right).\tag{2.28}$$

Integrate both sides of (2.28). We obtain that for r sufficiently large,

$$\begin{aligned}\phi_1(r)&=\frac{e^{\alpha_0 r}}{r^{N-1}}\int_r^\infty s^{N-1}e^{-\alpha_0 s}\left(-\frac{(N-1)\,\phi_\infty}{s}+\frac{\mu}{s^p}+O\left(\frac{1}{s^2}\right)\right)\\&=\frac{(N-1)\phi_\infty}{\alpha_0 r}-\frac{\mu}{\alpha_0 r^p}+O\left(\frac{1}{r^2}\right).\end{aligned}$$

Therefore we have that

$$\phi_1(r) = \begin{cases} \dfrac{(N-1)\phi_\infty}{\alpha_0 r} - \dfrac{\mu}{\alpha_0 r^p} + O\left(\dfrac{1}{r^2}\right) & \text{if } 1 < p < 2, \\ \dfrac{(N-1)\phi_\infty}{\alpha_0 r} + O\left(\dfrac{1}{r^2}\right) & \text{if } p \geq 2. \end{cases} \tag{2.29}$$

Note that if $1 < p < 2$, the proof of Theorem 2.1 is finished.

Suppose now $p \geq 2$. Let $\phi_2(r) = \phi_1(r) - \frac{(N-1)\phi_\infty}{\alpha_0 r}$. Then $\phi_2(r) = O(r^{-2})$ as $r \to \infty$. By the Taylor expansion of function F we have

$$\begin{aligned} F(\phi_1) - F(0) - F'(0)\phi_1 &= \frac{1}{2}F''(0)\phi_1^2 + O(\phi_1^3) \\ &= \frac{F''(0)c_1^2}{2r^2} + O\left(\frac{1}{r^3}\right), \end{aligned}$$

where $c_1 = (N-1)\phi_\infty/\alpha_0$. Thus by (2.21) and (2.22), it follows that

$$\phi_2' - \alpha_0\phi_2 + \frac{N-1}{r}\phi_2 = \frac{\tilde{c}_2^2}{r^2} + \frac{\mu}{r^p} + O\left(\frac{1}{r^3}\right),$$

where

$$\tilde{c}_2 = \frac{F''(0)c_1^2}{2} - (N-2)c_1.$$

We can then repeat the same process to obtain the expansion of ϕ_2 and furthermore the expansion as stated in Theorem 2.1 to any polynomial order as we want.

Next we need to determine c_i ($i \geq 0$) in Theorem 2.1. By (2.29), we already obtain the expansion in the case when $1 < p < 2$. In general, let k be the integer such that $k \leq p < k+1$. By the Taylor expansion of the function $F(t)$ at $t = 0$, we obtain that

$$\phi_1' - \alpha_0\phi_1 + \frac{N-1}{r}\phi_1 = -\frac{N-1}{r}\phi_\infty + \frac{\mu}{r^p} + \sum_{n=2}^{k} \frac{F^{(n)}(0)}{n!}\phi_1^n(r) + O(\phi_1^{k+1}) + O(u^\delta). \tag{2.30}$$

Let

$$\phi_1 = \sum_{i=1}^{k} \frac{c_i}{r^i} + \frac{d_1}{r^p} + O\left(\frac{1}{r^{k+1}}\right),$$

Substituting ϕ_1 into equation (2.30), we get by comparing the coefficients of r^{-l} ($l = 1, 2, \cdots, k$) that

$$c_1 = \frac{(N-1)\phi_\infty}{\alpha_0},$$

and that $\{c_i\}_{i=2}^k$ and d_1 are determined uniquely by

$$(N-i)c_{i-1} - \alpha_0 c_i = \sum_{n=2}^{i} \frac{F^{(n)}(0)}{n!} \sum_{\substack{j_1+\cdots+j_n=i \\ j_1,\cdots,j_n>0}} c_{j_1}c_{j_2}\cdots c_{j_n}, \quad \text{and} \quad d_1 = -\frac{\mu}{\alpha_0}.$$

The proof of Theorem 2.1 is complete. □

Now we prove Theorem 1.2.

Proof of Theorem 1.2. Let c_0, c_1 be defined as in Theorem 2.1. We have by (2.12), for $1 < p < 2$, that

$$\begin{aligned} \frac{u'}{u} &= -c_0^{\frac{1}{p-1}}\left(1 + \frac{c_1}{(p-1)c_0}\frac{1}{r} + O\left(\frac{1}{r^p}\right)\right) \\ &= -\left(\frac{m}{p-1}\right)^{\frac{1}{p}} - \frac{N-1}{p(p-1)}\frac{1}{r} + O\left(\frac{1}{r^p}\right); \end{aligned}$$

and for $p \geq 2$, that

$$\begin{aligned}\frac{u'}{u} &= -c_0^{\frac{1}{p-1}}\left(1+\frac{c_1}{(p-1)c_0}\frac{1}{r}+O\left(\frac{1}{r^2}\right)\right)\\ &= -\left(\frac{m}{p-1}\right)^{\frac{1}{p}}-\frac{N-1}{p(p-1)}\frac{1}{r}+O\left(\frac{1}{r^2}\right).\end{aligned}$$

It follows easily from the above equations that

$$\lim_{|x|\to\infty} u(x)|x|^{\frac{N-1}{p(p-1)}}e^{\left(\frac{m}{p-1}\right)^{\frac{1}{p}}|x|}=C$$

for a constant $0<C<\infty$. The proof of Theorem 1.2 is complete. □

3. Proof of Theorem 1.3

In this section, we prove Theorem 1.3. We need the following estimate.

Proposition 3.1. *Assume that $m>0$, $0\leq\mu<\bar{\mu}=((N-p)/p)^p$ and that $f\in C(\mathbb{R})$ satisfies (1.2) and (1.3). Let $u\in W^{1,p}(\mathbb{R}^N)$ be a weak solution to equation (1.1). Then there exists a positive constant c depending on N,p,μ,m,q,A and u such that*

$$|u(x)|\leq c|x|^{-\frac{N-p}{p}+\tau_0} \qquad \text{for } |x|<r_0,$$

where τ_0 and r_0 are constants in $(0,1)$ depending on N,p,μ,m,q,A and u.

The same estimate was obtained in [12, Proposition 2.1] for solutions to equation

$$-\Delta_p u-\frac{\mu}{|x|^p}|u|^{p-2}u=h(x)|u|^{p^*-2}u \qquad \text{in } \mathbb{R}^N,$$

where h is a bounded function. The proof of Proposition 3.1 is the same as that of Proposition 2.1 of [12], with minor modifications. We omit the details.

Now we prove Theorem 1.3. For simplicity, we write $B_r=B_r(0)$ in this section.

Proof of Theorem 1.3. Let $u\in W^{1,p}(\mathbb{R}^N)$ be a solution to equation (1.1). We prove (1.13) of Theorem 1.3 by Theorem 1.4. Set

$$g(x)=-m+\frac{f(u(x))}{|u(x)|^{p-2}u(x)}.$$

Then $u\in W^{1,p}(B_1)$ is a weak solution to equation (1.16) in B_1 with function g defined as above. By (1.2) and (1.3), we have

$$|g(x)|\leq c(1+|u(x)|^{p^*-p}).$$

Then by Proposition 3.1, we have

$$|g(x)|\leq c|x|^{-\alpha} \qquad \text{for } |x|<r_0,$$

where $\alpha=(p^*-p)(\frac{N-p}{p}-\tau_0)<p$ and τ_0,r_0 are as in Proposition 3.1. Thus Theorem 1.4 implies that

$$u(x)\leq c_1|x|^{-\gamma_1} \qquad \text{for } |x|<r_1,$$

where c_1, r_1 are constants and $r_1\leq r_0$. We can also prove the above estimate for $-u$ similarly. Thus (1.13) is proved.

Next, we prove (1.14). Suppose that u and $f(u)$ are nonnegative in B_ρ for $\rho>0$. Then u is a nonnegative supersolution to equation

$$-\Delta_p w-\frac{\mu}{|x|^p}|w|^{p-2}w=-m|w|^{p-2}w, \tag{3.1}$$

in B_ρ. We will construct a weak subsolution $v \in W^{1,p}(B_{r_2})$ to equation (3.1) in B_{r_2} for some $r_2 \le \rho$ such that $v \le u$ on ∂B_{r_2} and $v \ge c_2\Big(\inf_{B_{r_2}} u\Big)|x|^{-\gamma_1}$ in B_{r_2}. Then we obtain (1.14) by applying Theorem 1.5 to the supersolution u and the subsolution v of equation (3.1) in B_{r_2}.

In the rest of the proof, we construct such a subsolution v. We follow [12] and define $w_0(x) = |x|^{-\gamma_1}(1+\delta|x|^\epsilon)$ for some constants $\delta, \epsilon > 0$ to be determined. Direct computation shows that $w_0 \in W^{1,p}(B_1)$ solves the equation

$$-\Delta_p w - \frac{\mu}{|x|^p}|w|^{p-2}w = \frac{h(-\delta|x|^\epsilon)}{(1+\delta|x|^\epsilon)^{p-1}\,|x|^p}|w|^{p-2}w \qquad \text{for } x \neq 0, \tag{3.2}$$

where

$$h(t) \equiv |\gamma_1 - (\gamma_1-\epsilon)t|^{p-2}[k(\gamma_1-\epsilon)t - k(\gamma_1)] - \mu|1-t|^{p-2}(1-t), \quad t \in \mathbb{R},$$

and $k(t) \equiv (p-1)t^2 - (N-p)t$. Set

$$\tilde{h}(x) = \frac{h(-\delta|x|^\epsilon)}{(1+\delta|x|^\epsilon)^{p-1}\,|x|^p}, \quad x \in \mathbb{R}^N. \tag{3.3}$$

We want to choose appropriate δ, ϵ such that $\tilde{h}(x) \le -m$ for $|x|$ small enough.

Note that $h(0) = -\gamma_1^{p-2}k(\gamma_1) - \mu$, where $k(\gamma_1) = (p-1)\gamma_1^2 - (N-p)\gamma_1$. Thus by the definition of γ_1, as in (1.12), we have $h(0) = 0$. We also have

$$h'(0) = (p-1)\gamma_1^{p-2}(-p\gamma_1 + N - p + \epsilon)\epsilon > 0,$$

since $\gamma_1 < (N-p)/p$. Therefore there exists $1 > \delta_h > 0$ such that

$$2h'(0)t \le h(t) \le \frac{1}{2}h'(0)t \qquad \text{for } -\delta_h \le t < 0. \tag{3.4}$$

Now we choose $\delta = \delta_h$ and $0 < \epsilon < p$. Note that $1 + \delta|x|^\epsilon \ge 1$. Hence by (3.3) and (3.4) we have

$$-2h'(0)\delta_h|x|^{\epsilon-p} \le \tilde{h}(x) \le -\frac{1}{2}h'(0)\delta_h|x|^{\epsilon-p} \qquad \text{for } |x| < 1. \tag{3.5}$$

Since $\epsilon > 0$, (3.5) implies that $\tilde{h} \in L^{\frac{N}{p}}(B_1)$. Also it is clear that one can find a constant r_2, $0 < r_2 < \rho$, such that

$$\tilde{h}(x) \le -\frac{1}{2}h'(0)\delta_h|x|^{\epsilon-p} \le -m \qquad \text{for } |x| < r_2.$$

Hence w_1 is a weak subsolution to equation (3.1) in B_{r_2}.

For such w_0 and r_2, we define $v(x) = c'lw_0(x)$ for $x \in B_{r_2}$, where $c' = \inf_{\partial B_{r_2}} w_0^{-1}$ and $l = \inf_{B_{r_2}} u$. We can assume that $\inf_{B_{r_2}} u > 0$. Otherwise, (1.14) is trivial since we assume that $u \ge 0$. Thus $v \in W^{1,p}(B_{r_2})$ is a subsolution to equation (3.1) in B_{r_2} satisfying $v \le u$ on ∂B_{r_2} and $v \ge c_2\Big(\inf_{B_{r_2}} u\Big)|x|^{-\gamma_1}$ in B_{r_2}. We finish the proof. $\square$

We remark here that the proof for Theorem 1.3 also works for all $m \in \mathbb{R}$.

4. Proof of Theorem 1.6

In this section we prove Theorem 1.6. We need the following lemma. For simplicity, we write $B_\rho^c = B_\rho^c(0)$ in this section.

Lemma 4.1. *(i) Let $\alpha = ((m-\epsilon)/(p-1))^{1/p}$ for $m > \epsilon > 0$. Then the function $w_1(x) = e^{-\alpha|x|}$ is a solution to equation*

$$-L_{p,m-\epsilon}w \equiv -\Delta_p w + (m-\epsilon)|w|^{p-2}w = \frac{(N-1)\alpha^{p-1}}{|x|}|w|^{p-2}w \qquad \text{in } \mathbb{R}^N. \tag{4.1}$$

(ii) Let $\gamma \in \mathbb{R}$, $0 < \delta < 1/2$ and let

$$v_\gamma(x) = |x|^{-\frac{N-1}{p(p-1)}} e^{-\left(\frac{m}{p-1}\right)^{\frac{1}{p}}|x|} \left(1 - \gamma |x|^{-\delta}\right), \qquad x \neq 0.$$

Then v_γ is a solution to equation

$$-L_{p,m} v \equiv -\Delta_p v + m|v|^{p-2}v = Q(x)|v|^{p-2}v \qquad \text{in } \mathbb{R}^N, \tag{4.2}$$

where $Q(x)$ satisfies

$$Q(x) = \frac{Q_0}{|x|^{\delta+1}} + O\left(\frac{1}{|x|^{2\delta+1}}\right) \qquad \text{as } |x| \to \infty, \tag{4.3}$$

with $Q_0 = \left(\frac{m}{p-1}\right)^{\frac{p-1}{p}} p(p-1)\delta\gamma$.

Proof. We prove Lemma 4.1 by direct computation. First, we prove (i). Let $w_1 = e^{-\alpha|x|}$. Then

$$-L_{p,m-\epsilon} w_1 = -\left(|w_1'(r)|^{p-2} w_1'(r)\right)' - \frac{N-1}{r}|w_1'(r)|^{p-2}w_1'(r) + (m-\epsilon)w_1^{p-1}(r)$$

for $r = |x|$. Since $w_1' = -\alpha w_1$, we have that

$$|w_1'(r)|^{p-2}w_1'(r) = -\alpha^{p-1}w_1^{p-1}(r),$$
$$\left(|w_1'(r)|^{p-2}w_1'(r)\right)' = (m-\epsilon)w_1^{p-1}(r).$$

Hence

$$-L_{p,m-\epsilon}w_1 = \frac{(N-1)\alpha^{p-1}}{r}w_1^{p-1}(r).$$

This proves (i).

Next, we prove (ii). Write $\alpha = \frac{N-1}{p(p-1)}$, $\beta = \left(\frac{m}{p-1}\right)^{\frac{1}{p}}$ and set

$$v_\gamma(r) = e^{-\beta r} r^{-\alpha}(1 - \gamma r^{-\delta})$$

for $r = |x|$. Then $v_\gamma(r) > 0$ for r large enough and

$$-L_{p,m}v_\gamma = -\left(|v_\gamma'(r)|^{p-2}v_\gamma'(r)\right)' - \frac{N-1}{r}|v_\gamma'(r)|^{p-2}v_\gamma'(r) + mv_\gamma^{p-1}(r).$$

We have $v_\gamma'(r) = -A(r)v_\gamma$, where

$$A(r) = \beta + \frac{\alpha}{r} + \frac{\delta\gamma r^{-\delta-1}}{1-\gamma r^{-\delta}}.$$

Note that $A(r) > 0$ for r large enough. We also have that

$$|v_\gamma'(r)|^{p-2}v_\gamma'(r) = -A^{p-1}(r)v_\gamma^{p-1}(r),$$
$$\left(|v_\gamma'(r)|^{p-2}v_\gamma'(r)\right)' = -\left((A^{p-1}(r))' - (p-1)A^p(r)\right)v_\gamma^{p-1}(r).$$

Hence

$$-L_{p,m}v_\gamma = \left(m + (A^{p-1}(r))' - (p-1)A^p(r) + \frac{N-1}{r}A^{p-1}(r)\right)v_\gamma^{p-1}(r).$$

Thus (4.2) is proved by setting

$$Q(x) = Q(r) = m + (A^{p-1}(r))' - (p-1)A^p(r) + \frac{N-1}{r}A^{p-1}(r)$$

for $r = |x|$. We need to show that Q satisfies (4.3). To this end, we have

$$
\begin{aligned}
A(r) &= \beta + \frac{\alpha}{r} + \frac{\delta\gamma}{r^{\delta+1}} + O\left(\frac{1}{r^{2\delta+1}}\right),\\
A^{p-1}(r) &= \beta^{p-1}\left(1 + \frac{(p-1)\alpha}{\beta r} + \frac{(p-1)\delta\gamma}{\beta r^{\delta+1}} + O\left(\frac{1}{r^{2\delta+1}}\right)\right),\\
A^{p}(r) &= \beta^{p}\left(1 + \frac{p\alpha}{\beta r} + \frac{p\delta\gamma}{\beta r^{\delta+1}} + O\left(\frac{1}{r^{2\delta+1}}\right)\right),\\
\left(A^{p-1}(r)\right)' &= -\frac{\beta^{p-2}(p-1)\alpha}{r^2} + O\left(\frac{1}{r^{\delta+2}}\right) = O\left(\frac{1}{r^2}\right),
\end{aligned}
$$

as $r \to \infty$. Then (4.3) follows easily. The proof of (ii) is complete. □

Now we prove Theorem 1.6.

Proof of Theorem 1.6. Let $u \in W^{1,p}(\mathbb{R}^N)$ be a solution to equation (1.1). We claim that there exist $\alpha > 0$ and $C > 0$ such that for ρ large enough we have

$$
|u(x)| \le Ce^{-\alpha|x|} \qquad \text{for } |x| \ge \rho. \tag{4.4}
$$

To prove (4.4), we can follow the argument of [6, Theorem 1.1] to prove that $u \in C^1(\mathbb{R}^N\backslash\{0\})$ and $u(x) \to 0$ as $|x| \to \infty$. And then by (1.2), we obtain that

$$
\frac{|f(u(x))|}{|u(x)|^{p-1}} \le C|u(x)|^{q-p} \to 0 \qquad \text{as } |x| \to \infty. \tag{4.5}
$$

Fix $\epsilon > 0$ such that $0 < \epsilon < m$. We can choose ρ_0 large enough such that $|f(u(x))|/|u(x)|^{p-1} \le \epsilon$ for $|x| \ge \rho_0$. Then u is a subsolution to equation

$$
-\Delta_p w + (m-\epsilon)|w|^{p-2}w = \frac{\mu}{|x|^p}|w|^{p-2}w \tag{4.6}
$$

in $B^c_{\rho_0}$.

Let $\alpha = ((m-\epsilon)/(p-1))^{1/p}$ and set $w_1 = e^{-\alpha|x|}$. Then by Lemma 4.1 (i) w_1 is a solution to equation (4.1). We can choose $\rho \ge \rho_0$ large enough such that

$$
\frac{(N-1)\alpha^{p-1}}{|x|} \ge \frac{\mu}{|x|^p} \qquad \text{for } |x| \ge \rho.
$$

Meanwhile, we can also choose $\rho \ge \rho_0$ large enough for later use such that

$$
m - \epsilon - \frac{\mu}{|x|^p} > 0 \qquad \text{for } |x| \ge \rho.
$$

Then w_1 is a supersolution to equation (4.6) in B^c_ρ. Now define $\tilde{w}_1(x) = CMw_1(x)$, where $C = e^{\alpha\rho}$ and $M = \sup_{\partial B_\rho} u^+$. Then $\tilde{w}_1$ is also a supersolution to equation (4.6) in B^c_ρ and $u \le \tilde{w}_1$ on ∂B_ρ.

Let $(u-\tilde{w}_1)^+ = \max(u-\tilde{w}_1, 0)$. Since u is a subsolution to equation (4.6) in B^c_ρ and $\tilde{w}_1$ is a supersolution to equation (4.6) in B^c_ρ respectively, we have that

$$
\int_{B^c_\rho} |\nabla u|^{p-2}\nabla u \cdot \nabla(u-\tilde{w}_1)^+ + \int_{B^c_\rho}\left(m-\epsilon-\frac{\mu}{|x|^p}\right)|u|^{p-2}u(u-\tilde{w}_1)^+ \le 0, \tag{4.7}
$$

and that

$$
\int_{B^c_\rho} |\nabla \tilde{w}_1|^{p-2}\nabla \tilde{w}_1 \cdot \nabla(u-\tilde{w}_1)^+ + \int_{B^c_\rho}\left(m-\epsilon-\frac{\mu}{|x|^p}\right)|\tilde{w}_1|^{p-2}\tilde{w}_1(u-\tilde{w}_1)^+ \ge 0. \tag{4.8}
$$

Then combining (4.7) and (4.8) yields

$$\begin{aligned}&\int_{B_\rho^c}\langle|\nabla u|^{p-2}\nabla u-|\nabla\tilde{w}_1|^{p-2}\nabla\tilde{w}_1,\nabla(u-\tilde{w}_1)^+\rangle\\&+\int_{B_\rho^c}\left(m-\epsilon-\frac{\mu}{|x|^p}\right)\left(|u|^{p-2}u-|\tilde{w}_1|^{p-2}\tilde{w}_1\right)(u-\tilde{w}_1)^+\le 0.\end{aligned}\tag{4.9}$$

Then it follows easily from (4.9) that $u\le\tilde{w}_1$ in B_ρ^c. We can prove similarly that $-u\le\tilde{w}_1$ in B_ρ^c. This proves (4.4).

Now we prove (1.21). We only prove that

$$u(x)\le C_1|x|^{-\frac{N-1}{p(p-1)}}e^{-\left(\frac{m}{p-1}\right)^{\frac{1}{p}}|x|}\qquad\text{for } |x|>R_1,\tag{4.10}$$

where R_1 is a constant large enough. We can prove similarly the same estimate for $-u$.

Let

$$c(x)=\frac{\mu}{|x|^p}+\frac{f(u(x))}{|u(x)|^{p-2}u(x)}.$$

Then u satisfies that

$$-\Delta_p u+m|u|^{p-2}u=c(x)|u|^{p-2}u$$

in $\mathbb{R}^N$. By (4.4) and (4.5), we have that

$$|c(x)|\le\frac{2\mu}{|x|^p}<m\qquad\text{for } |x|\ge\rho_1$$

where ρ_1 is a constant large enough. Thus u is a subsolution to equation

$$-\Delta_p w+m|w|^{p-2}w=\frac{2\mu}{|x|^p}|w|^{p-2}w,\tag{4.11}$$

in $B_{\rho_1}^c$.

Let

$$v_1(x)=|x|^{-\frac{N-1}{p(p-1)}}e^{-\left(\frac{m}{p-1}\right)^{\frac{1}{p}}|x|}\left(1-|x|^{-\delta}\right),\qquad x\ne 0,$$

where $0<\delta<\min(p-1,1/2)$. By Lemma 4.1 (ii), v_1 is a solution to equation (4.2) with

$$Q(x)=\frac{Q_0}{|x|^{\delta+1}}+O\left(\frac{1}{|x|^{2\delta+1}}\right)\qquad\text{as } |x|\to\infty,$$

where

$$Q_0=\left(\frac{m}{p-1}\right)^{\frac{p-1}{p}}p(p-1)\delta>0.$$

Since $\delta<p-1$, we have that

$$Q(x)\ge\frac{2\mu}{|x|^p}\qquad\text{for } |x|\ge R_1,$$

where R_1, $R_1\ge\rho_1$, is a constant. Hence v_1 is a supersolution to equation (4.11) in $B_{R_1}^c$. Now define $\tilde{v}_1(x)=CMv_1(x)$, where $C=\sup_{\partial B_{R_1}}v_1^{-1}$ and $M=\sup_{\partial B_{R_1}}u^+$. Then $\tilde{v}_1$ is also a supersolution to equation (4.11) in $B_{R_1}^c$ and $u\le\tilde{v}_1$ on ∂B_{R_1}. By the same argument as above, we can easily obtain that

$$u(x)\le\tilde{v}_1(x)\qquad\text{for } |x|\ge R_1.$$

This proves (4.10). Similarly we can prove the same estimate for $-u$. So (1.21) is proved.

We prove (1.22) similarly. Suppose that both u and $f(u)$ are nonnegative in B_ρ^c for $\rho>1$. Then u is a nonnegative supersolution of equation

$$-\Delta_p w+m|w|^{p-2}w=0\tag{4.12}$$

in B_ρ^c.

Let

$$v_{-1}(x) = |x|^{-\frac{N-1}{p(p-1)}} e^{-\left(\frac{m}{p-1}\right)^{\frac{1}{p}}|x|}\left(1+|x|^{-\delta}\right), \qquad x \neq 0,$$

where $0<\delta<1/2$. By Lemma 4.1 (ii), v_{-1} is a solution to equation (4.2) with

$$Q(x) = \frac{Q_0}{|x|^{\delta+1}} + O\left(\frac{1}{|x|^{2\delta+1}}\right) \qquad \text{as } |x| \to \infty,$$

where

$$Q_0 = -\left(\frac{m}{p-1}\right)^{\frac{p-1}{p}} p(p-1)\delta < 0. \tag{4.13}$$

It follows from (4.13) that

$$Q(x) \le 0 \qquad \text{for } |x| > R_2,$$

where R_2, $R_2 > \rho$, is a large constant. Hence v_{-1} is a subsolution to equation (4.12) in $B_{R_2}^c$. Now define $\tilde{v}_{-1} = C_2 l' v_{-1}$, where $C_2 = \inf_{\partial B_{R_2}} v_2^{-1}$ and $l' = \inf_{\partial B_{R_2}} u$. Then $\tilde{v}_{-1}$ is also a subsolution to equation (4.12) in $B_{R_2}^c$ and $\tilde{v}_{-1} \le u$ on ∂B_{R_2}. By the same argument as above, we can easily obtain that

$$\tilde{v}_{-1} \le u \qquad \text{in } B_{R_2}^c.$$

This proves (1.22). The proof of Theorem 1.6 is complete. □

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